Harmonic potential driven by long-range correlated noise

Manuel O. Cáceres*
Centro Atómico Bariloche and Instituto Balseiro CNEA and Universidad Nacional de Cuyo, Av. Ezequiel Bustillo Km 9.5, 8400 San Carlos de Bariloche, Río Negro, Argentina

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The probability distribution of a particle in a quadratic potential driven by Gaussian long-range correlated noise has been obtained. The long-time asymptotic relaxation of the stochastic process has been characterized in terms of the long-range correlated noise appearing in the corresponding stochastic differential equation defining the process. The particular case when the particle is driven by a Gaussian color noise has also been revisited, in this way giving its exact probability distribution for all time. By using a characteristic functional technique reported previously, results for a non-Gaussian long-range correlated noise are also shown. Emphasis has been placed on solving the plane rotator in the presence of arbitrary random torques having long-range correlations. In order to show strong and weak non-Markovian effects coming from different sources of noise, the undamped free particle under the influence of arbitrary accelerations has been analyzed. We also analyze in detail the structure of the trajectories of the overdamped and undamped free particle.

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I. INTRODUCTION

The long-range noise problem is connected with the statistical properties of a strong non-Markovian stochastic process (SP) driven by that noise, i.e., its dynamics never match the Markovian dynamics. Consider, for example, the displacement $X(t)$ of a particle under the influence of an harmonic potential $U(X)=\frac{1}{2}\gamma X^2$ and a stochastic term $\xi(t)$; its equation of motion is governed by the stochastic differential equation (SDE)

$$\frac{dX}{dt}=-\gamma X+\xi(t), \quad \gamma\geq 0, \quad X\in(-\infty,\infty). \tag{1.1}$$

If the noise term $\xi(t)$ is a zero-mean Gaussian SP with a long-range stationary correlation, say[1]

$$\langle\xi(t_1)\xi(t_2)\rangle=\frac{\Gamma_2\tau^{-1}}{(1+|t_1-t_2|/\tau)^{\mu}}, \quad \tau>0;\mu\geq 0, \tag{1.2}$$

the displacement of the particle (if $0\leq\mu<1$) will show a strong non-Markovian character, i.e., its dynamical properties—even in its asymptotic regime—will not share the behavior of the usual overdamped Brownian motion. [We will be interested in finite τ and in the range of $\mu\in[0,1)$; this is so because previous analysis of the generalized Wiener process has shown that a strong non-Markovian behavior (superdiffusion) is only obtained if $\mu<1$. The case $\mu>1$ produces weak non-Markovian effects, and the case $\mu=1$ has logarithmic corrections.] This is not the case if the noise is of the short-range type [or if $\mu>1$ in Eq. (1.2)]. In this case we say that we are in the presence of a weak non-Markovian SP $X(t)$, i.e., after a long transient the dynamical properties of $X(t)$ are equivalent (renormalized) with respect to the Markovian model [1].

The simplest strong non-Markovian model is when $\gamma=0$; in this limit Eq. (1.1) goes to the generalized Wiener process [2], and its statistical properties have been shown to share some aspects of the persistent fractional Brownian motion (fBm) [3]. This fact follows from studying the fractal structure of the realization [4] of the SP $X(t)$ (see Appendix A for a short review). Other strong non-Markovian processes are those related to second order SDE's when the noise term [the random force $\xi(t)$] is of the long-range type of Eq. (1.2). In the present paper the second order SDE $(d^2/dt^2)\phi(t)+\gamma(d/dt)\phi=\xi(t)$ will be used to study a random phase problem (see Sec. III A). A similar situation occurs when studying the evolution equation of an undamped free particle, i.e., an equation of motion like $(d^2/dt^2)Y(t)=\xi(t)$. This problem can also be worked out using our functional approach for any kind of noise (Gaussian or not; see Appendix B). In particular, for a long-range Gaussian noise we show in Sec. III B that the critical value $\mu=1$ is also the threshold between the strong and weak non-Markovian behavior for the undamped free particle.

The analysis of weak non-Markovian stochastic processes has been done using different techniques [5–7], among which path integral formulations have been very successful for treating color noise in arbitrary potentials [8]. However, the path integral technique does not seem to be very useful for studying some of the issues appearing in strong non-Markovian processes. The literature on weak non-Markovian problems is quite vast, but long-range noise problems have not been solved due to the intrinsic difficulties of the perturbation theory; only recently Masoliver and Wang [9] studied the probability density functions associated with linear Langevin-like equations having a long-range Gaussian noise.

*Senior Independent Research Associate at CONICET. Electronic address: caceres@cab.cnea.gov.ar

[In that reference, Eq. (4.2), the problem has been worked out by using the Tauberian theorem; therefore, in order to fulfill the hypothesis of that theorem the values of $\alpha=\mu$ are restricted to $\mu\in(1,2)$. This is not the case in the present paper because we have carried out explicit integrations.] Using our functional approach, we can go one step further, and study (in an exact way) cosine correlations associated with the random phase problem. In the present paper we give a detailed discussion on the strong non-Markovian problem using the advantages of some exact functional results previously reported [2]. As an additional bonus we can trivially work out the Gaussian color-noise (short-range) problem for the harmonic potential.

We are aware that our method works at least for the quadratic potential because we already have the exact generating functional [2] of the SP $X(t)$. On the other hand, our approach can also be used to work out arbitrary linear stochastic differential equations, autonomous or not, with or without inertia, and for any structure of noise having long-range correlations [2,10]. What is even more important is that using the present results we can also study some related bounded problems [11]. Hence our functional approach is an enlightening contribution to the analysis of strong non-Markovian effects on arbitrary linear SDE's. An application of the present approach is the study of the dipole correlation for a rigid rotator in the presence of Gaussian random torques [12,13,1] having long-range correlations. For this problem the calculation of the moments and/or correlations of the cosine of the angle is a nontrivial task which can be tackled using our functional approach [see, for example, solutions (3.4) and (3.8)].

It is well known that if a SP $X(t)$ is non-Markovian, its complete characterization demands knowledge of the whole Kolmogorov hierarchy, i.e., the m-time joint probability distribution $P[X(t_1);X(t_2);\ldots;X(t_m)]$, or equivalently all the m-time moments $\langle X(t_1)X(t_2)\cdots X(t_m)\rangle$ or cumulants $\langle\langle X(t_1)X(t_2)\cdots X(t_m)\rangle\rangle$. When partial knowledge of the SP is required, the one-time probability distribution $P[X(t_1)]$ is enough. This is the case when only one-time moments of the process $\langle X(t)^m\rangle$ are needed. In order to know the whole Kolmogorov hierarchy, knowledge of the characteristic functional $G_X([k(t)])=\langle\exp\int ik(t)X(t)dt\rangle$ is required, which of course is a much more complicated object than the one-time characteristic function [1]. The notation $G_X([k(t)])$ emphasizes that G depends on the whole test function $k(t)$, not just on the value it takes at one particular time t_j. The convergence of the integral is achieved because the functions $k(t)$ may be restricted to those that vanish for sufficiently large t. As a matter of fact, the possibility of having a closed expression for the noise characteristic functional allows us to find, by quadrature, the whole Kolmogorov hierarchy of the SP $X(t)$. We note that in this paper we do not make use of any partial differential equation, and that the results that we present are exact solutions for arbitrary noise.

II. A STRONG NON-MARKOVIAN GAUSSIAN PROCESS

The equation of motion for the position of a one-dimensional overdamped Brownian particle in a quadratic potential and in a generalized medium is characterized by Eq. (1.1). There $\xi(t)\in$ Re is the time-dependent random force representing the medium (the noise). When $\xi(t)$ is a zero-mean Gaussian white noise, i.e., $\langle\xi(s_1)\xi(s_2)\rangle=\Gamma_2\delta(s_1-s_2)$, Eq. (1.1) gives rise to the usual Fokker-Planck dynamics [1]. If the random force were a short-range noise, characterized by the stationary correlation

$$\langle\xi(t_1)\xi(t_2)\rangle=\frac{\Gamma_2}{2\theta}\exp(-|t_1-t_2|/\theta),\quad \theta\geqslant 0 \tag{2.1}$$

the dynamics of the particle would be driven by a color noise, which has been studied extensively in the past [5–8].

In the present paper we will be interested in the results coming from the long-range correlation (1.2). Thus let us allow the possibility of having an arbitrary zero-mean Gaussian noise with correlation $\langle\xi(t_1)\xi(t_2)\rangle$. (For a non-Gaussian noise with a long-range correlation, use, for example, a Campbell noise with a power-law shape pulse [2].) Note that a white-noise force can be reobtained from (2.1) in the limit $\theta\to 0$, while the ballistic case can be obtained from Eq. (1.2) in the limit $\mu\to 0$. [Only in the simultaneous limits $\mu\to 0$ and $\tau\to\infty$ is the long-range correlation (1.2) proportional to the long-correlated limit $\theta\to\infty$ of the short-range model (2.1); this fact shows the difference between both correlation functions. Analysis concerning this type of correlation has been applied to the study of a diffusion-advection equation with a long-range random velocity field [14].]

Let us consider the noise $\xi(t)$, with $t\in[0,\infty]$, to be a zero-mean Gaussian noise of intensity Γ_2 and characterized by some correlation. Its functional will be

$$G_\xi([k(t)])=\exp\left(\frac{-1}{2}\int_0^\infty\int_0^\infty k(s_1)k(s_2)\times\langle\xi(s_1)\xi(s_2)\rangle ds_1 ds_2\right), \tag{2.2}$$

where the correlation $\langle\xi(s_1)\xi(s_2)\rangle$ could be given by Eqs. (1.2) or (2.1), or any other suitable function.

By using proposition 1 of Ref. [2], it follows that the characteristic functional of the SP $X(t)$ is, for any noise $\xi(t)$, given by

$$G_x([Z(t)])=e^{+ik_0X_0}G_\xi\left(\left[\int_t^\infty e^{\gamma(t-s)}Z(s)ds\right]\right), \tag{2.3}$$

where X_0 is the initial condition of the SP $X(t)$ and k_0 is a functional of $Z(t)$ given by

$$k_0=\int_0^\infty e^{-\gamma s}Z(s)ds. \tag{2.4}$$

For the Gaussian case using Eq. (2.2) we obtain

$$G_X([Z(t)])=e^{+ik_0X_0}\exp\left[\frac{-1}{2}\int_0^\infty\int_0^\infty\left(\int_{s_1}^\infty e^{\gamma(s_1-s)}Z(s)ds\right)\times\left(\int_{s_2}^\infty e^{\gamma(s_2-s')}Z(s')ds'\right)\times\langle\xi(s_1)\xi(s_2)\rangle ds_1 ds_2\right]. \tag{2.5}$$

A non-Gaussian case is sketched in Appendix B. Note that for any structure of noise, all m-time moments of the SP $X(t)$ follow from mth-order functional differentiation of $G_X([Z(t)])$, i.e.,

$$\langle X(t_1)X(t_2)\cdots X(t_m)\rangle = i^{-m}\frac{\delta}{\delta Z(t_1)}\cdots\frac{\delta}{\delta Z(t_m)}G_X([Z(t)])\bigg|_{Z=0}. \tag{2.6}$$

On the other hand, the Kolmogorov hierarchy can be obtained by quadrature from the functional. In general, we can invert the characteristic functional by introducing the n-dimensional Fourier transform

$$\begin{aligned} P(X[t_1];X[t_2];\ldots;X[t_n]) &\equiv P(x_1,t_1;x_2,t_2;\ldots;x_n,t_n) \\ &= \frac{1}{(2\pi)^n}\int\cdots\int dk_1\cdots dk_n \exp\left(-i\sum_{i=1}^{n}k_i x_i\right) \\ &\quad\times[G_X([Z(t)])]_{Z(t)=k_1\delta(t-t_1)+\cdots+k_n\delta(t-t_n)}. \end{aligned} \tag{2.7}$$

Note that Eq. (2.3) is a result which allows us to obtain a complete characterization of a non-Markovian Gaussian SP $X(t)$. For example, the one-time probability distribution $P(X[t_1])\equiv P(x_1,t_1)$ is just given in terms of $G_X([Z(t)])$ evaluated with the test function $Z(s)=k_1\delta(s-t_1)$. From Eqs. (2.5) and (2.7) we see that to know the one-time probability distribution, we must calculate the integral

$$\sigma(t_1)\equiv\frac{1}{2}\int_0^{t_1}\int_0^{t_1}e^{\gamma(s_1-t_1)}e^{\gamma(s_2-t_1)}\langle\xi(s_1)\xi(s_2)\rangle ds_1 ds_2. \tag{2.8}$$

Thus the one-time (conditional) probability distribution $P(x,t)$ reads

$$\begin{aligned} P(x,t) &= \frac{1}{2\pi}\int e^{-ikx}\exp(-\sigma(t)k^2+iB(t)k)dk \\ &= 1/\sqrt{4\pi\sigma(t)}\exp[-(x-B(t))^2/4\sigma(t)]; \end{aligned} \tag{2.9}$$

in general, $\sigma(t)$ is given by Eq. (2.8) and $B(t)=X_0e^{-\gamma t}$. Therefore, for the Gaussian case the analysis of the stationary probability distribution, and the transient relaxation of the SP $X(t)$ is reduced to the study of the generalized dispersion $\sigma(t)$.

A. Calculation of $\sigma(t=\infty)$

Equation (2.8) can be written in a more suitable way, making easier the analysis of its stationary value and also of its asymptotic long-time behavior:

$$\begin{aligned} \sigma(t) = \frac{1}{2}\Bigg\{&\int_0^t ds_1\int_0^{s_1}e^{\gamma(s_1-t)}e^{\gamma(s_2-t)}\langle\xi(s_1)\xi(s_2)\rangle ds_2 \\ &+\int_0^t ds_1\int_{s_1}^t e^{\gamma(s_1-t)}e^{\gamma(s_2-t)}\langle\xi(s_1)\xi(s_2)\rangle ds_2\Bigg\}. \end{aligned} \tag{2.10}$$

Using the fact that the correlation $\langle\xi(s_1)\xi(s_2)\rangle$ is stationary, the generalized dispersion $\sigma(t)$ can be rewritten in the form

$$\begin{aligned} \sigma(t) = \frac{1}{2}\Bigg\{&\int_0^t e^{-2\gamma s_1}ds_1\int_0^{t-s_1}e^{-\gamma s_2}\langle\xi(s_2)\xi(0)\rangle ds_2 \\ &+\int_0^t e^{-\gamma s_1}ds_1\int_0^{s_1}e^{-\gamma(s_1-s_2)}\langle\xi(s_2)\xi(0)\rangle ds_2\Bigg\}. \end{aligned} \tag{2.11}$$

From this expression, and using the Laplace transform notation $\mathcal{L}_u[f(t)]\equiv\int_0^\infty e^{-uz}f(z)dz$, i.e., $z\to u$, it is possible to see that the stationary generalized dispersion is given by

$$\sigma(t=\infty) = \frac{1}{2}\left(\frac{1}{2\gamma}\mathcal{L}_\gamma[\langle\xi(z)\xi(0)\rangle] + \mathcal{L}_\gamma\left[\int_0^{s_1}e^{-\gamma(s_1-s_2)}\langle\xi(s_2)\xi(0)\rangle ds_2\right]\right). \tag{2.12}$$

Therefore, by using Laplace's convolution theorem, we finally arrive at the expression

$$\sigma_\infty\equiv\sigma(t=\infty)=\frac{1}{2\gamma}\mathcal{L}_\gamma[\langle\xi(z)\xi(0)\rangle]. \tag{2.13}$$

This is an exact result which gives the generalized dispersion of the stationary probability distribution of a non-Markovian Brownian particle in a harmonic potential, in the presence of an arbitrary correlated Gaussian noise; i.e., from Eq. (2.9) the stationary nonequilibrium probability distribution is

$$P_{st}(x)=\frac{1}{\sqrt{4\pi\sigma_\infty}}\exp\left(\frac{-x^2}{4\sigma_\infty}\right), \tag{2.14}$$

where the generalized stationary dispersion σ_∞ is given by Eq. (2.13).

In order to exemplify formula (2.13), let us first reobtain the familiar white-noise case (Fokker-Planck dynamics), and let us also compute the well studied color-noise problem (weak non-Markovian case) before analyzing the long-range case.

1. White-noise case

Using our approach this case is trivial because $\mathcal{L}_\gamma[\langle\xi(z)\xi(0)\rangle]=\mathcal{L}_\gamma[\Gamma_2\delta(z)]=\Gamma_2/2$. Therefore, the stationary generalized dispersion turns out to be $\sigma_\infty=\Gamma_2/4\gamma$, as can be expected from the corresponding Smoluchowski equation $\partial_t P(x,t)=[\partial_x\gamma x+(\Gamma_2/2)\partial_x^2]P(x,t)$.

2. Color-noise case

This is achieved by using correlation (2.1) in Eq. (2.13). First we calculate $\mathcal{L}_\gamma[\langle\xi(z)\xi(0)\rangle]=\mathcal{L}_\gamma[(\Gamma_2/2\theta)\exp(-z/\theta)]=(\Gamma_2/2)(\gamma\theta+1)^{-1}$. Therefore, the generalized stationary dispersion gives

$$\sigma_\infty=\frac{\Gamma_2}{4\gamma}(\gamma\theta+1)^{-1},\quad \gamma>0,\quad \theta\geqslant 0, \tag{2.15}$$

which of course is in agreement with reported results obtained by using other techniques [5–8], showing in this way the simplicity and elegance of our formula (2.13).

3. Long-range case

The case where the noise has long-range correlation has not been addressed before, to our knowledge. The important fact is that this case can also be analyzed by using correlation (1.2) in formula (2.13). First we have to calculate the Laplace transform of the long-range noise correlation function:

$$\mathcal{L}_\gamma[\langle\xi(z)\xi(0)\rangle]=\mathcal{L}_\gamma\left[\frac{\Gamma_2\tau^{-1}}{(1+z/\tau)^\mu}\right]=(\gamma\tau)^{\mu-1}\Gamma_2 e^{\gamma\tau}\Gamma(1-\mu,\gamma\tau); \tag{2.16}$$

here $\Gamma(\nu,x)$ is the incomplete gamma function [15]. Therefore, the stationary generalized dispersion reads

$$\sigma_\infty=\frac{\tau\Gamma_2}{2}(\gamma\tau)^{\mu-2}e^{\gamma\tau}\Gamma(1-\mu,\gamma\tau),\quad \gamma>0,\quad \mu\geqslant 0. \tag{2.17}$$

This formula shows that the most important parameter, in our long-range model, is the exponent μ (the noise correlation power-law parameter). From this result we see a remarkable difference compared to short-range models (weak non-Markovian effects). For fixed γ and τ, increasing the noise correlation, i.e., $\mu\to 0$, the stationary dispersion σ_∞ increases. This phenomenon is exactly opposite to that obtained from a short-range correlated noise, i.e., increasing θ in Eq. (2.15) predicts, for a fixed γ, a decreasing dispersion σ_∞. This result can be interpreted heuristically because if $\mu\in[0,1)$ the increment of the SP $X(t)$ shows a persistence in its corresponding generalized Wiener increments (similar to superdiffusion in the fBm), therefore leading to an increase in the dispersion σ_∞ (see Appendix A for a detailed discussion). For a fixed μ, the behavior of the generalized dispersion σ_∞ as a function of the friction parameter γ is the same for strong or weak non-Markovian models, i.e., reducing (or increasing) γ implies a corresponding increase (or reduction) of the stationary dispersion σ_∞.

Thus we can conclude that a long-range (correlated) noise induces strong non-Markovian effects on the SP $X(t)$, which change the behavior of the stationary dispersion σ_∞ as a function of the noise correlation. Of course the strong non-Markovian effect also changes the dynamics (relaxation) of SP $X(t)$; this issue will be examined in detail in the next sections.

B. Asymptotic long-time behavior of $\sigma(t)$

The long-time regime of Eq. (2.8) can be estimated from the first term in Eq. (2.11) if we multiply this result by a factor of 2. Therefore, the long-time limit of the dispersion can be written as

$$\begin{aligned}\sigma(t)&=\int_0^t e^{-2\gamma s_1}ds_1\int_0^{t-s_1} e^{-\gamma s_2}\langle\xi(s_2)\xi(0)\rangle ds_2\\&=t^2\int_0^1 e^{-2\gamma t z_1}dz_1\int_0^{1-z_1} e^{-\gamma t z_2}\langle\xi(tz_2)\xi(0)\rangle dz_2.\end{aligned} \tag{2.18}$$

The integral in dz_2, in Eq. (2.18), can be worked out exactly. If we use the long-range correlation (1.2), we obtain

$$\begin{aligned}&\Gamma_2\int_0^{t-z_1} e^{-\gamma t z_2}\frac{\tau^{-1}}{(1+tz_2/\tau)^\mu}\,dz_2\\&=\frac{e^{\gamma\tau}(\gamma\tau)^{\mu-1}\Gamma_2}{t}[\Gamma(1-\mu,\gamma\tau)\\&\quad-\Gamma(1-\mu,\gamma\tau+\gamma t(1-z_1))],\end{aligned} \tag{2.19}$$

where $\Gamma(\nu,Z)$ is the incomplete gamma function [16]. Using Eq. (2.19) in Eq. (2.18), we obtain two contributions: The first one gives

$$\begin{aligned}&t\Gamma_2 e^{\gamma\tau}(\gamma\tau)^{\mu-1}\Gamma(1-\mu,\gamma\tau)\int_0^1 e^{-2\gamma t z_1}dz_1\\&=\sigma_\infty(1-e^{-2\gamma\tau}),\end{aligned} \tag{2.20}$$

where σ_∞ is given in Eq. (2.17). The second contribution gives

$$-t\Gamma_2 e^{\gamma\tau}(\gamma\tau)^{\mu-1}\int_0^1 e^{-2\gamma t z_1}\Gamma(1-\mu,\gamma\tau+\gamma t(1-z_1))dz_1. \tag{2.21}$$

This integral can also be approximated in the long-time limit $t\to\infty$. Since $z_1\in[0,1]$ and, using the expansion of the incomplete gamma function, $\Gamma(\nu,Z)\sim e^{-Z}/Z^{1-\nu}$ in the limit of $Z\to\infty$, it is possible to see (if $0\leqslant\mu<1$) that Eq. (2.21) reduces to

$$\begin{aligned}&-t\Gamma_2 e^{\gamma\tau}(\gamma\tau)^{\mu-1}\int_0^1 e^{-2\gamma t z_1}\frac{\exp(-\gamma t(1-z_1))}{[\gamma t(1-z_1)]^\mu}\,dz_1\\&=-\frac{\Gamma_2}{\gamma}e^{\gamma(\tau-t)}(\gamma t)^{1-\mu}(\gamma\tau)^{\mu-1}\frac{\Gamma(1-\mu)}{\Gamma(2-\mu)}\\&\quad\times M(1,2-\mu,-\gamma t),\end{aligned} \tag{2.22}$$

where $M(a,c,z)$ is the Kummer confluent hypergeometric function [16]. Therefore, the long-time dispersion $\sigma(t)$ is given by the sum of the expressions (2.20) and (2.22). Using the power expansion of the Kummer function $M(c,a,-z)\sim[\Gamma(c)/\Gamma(c-a)](-z)^a$ for large z and $c-a\neq 0,-1,-2,\ldots$, for the long-time behavior we obtain

$$\sigma(t)\sim\sigma_\infty\left[1-\frac{2(\gamma t)^{2-\mu}e^{-\gamma t}}{\Gamma(1-\mu,\gamma\tau)}-e^{-2\gamma t}+O(e^{-2\gamma t})\right],$$
$$\gamma>0,\quad 0\leqslant\mu<1, \tag{2.23}$$

where $O(e^{-2\gamma\tau})$ means an order smaller than $e^{-2\gamma\tau}$. This result shows that the strong non-Markovian effect changes the relaxation of the particle in the harmonic well. If $\mu\in[0,1)$ the relaxation is nonexponential, and it is characterized by the law $\sim(\gamma t)^{2-\mu}e^{-\gamma t}$.

For long-range correlated models, it is also expected that similar anomalous behavior will occur in calculating more complex objects such as the correlation function $\langle\langle X(t_1)X(t_2)\rangle\rangle$, see, for example, Sec. III. Our approach gives us the possibility to do this from Eq. (2.6). In fact the analysis is reduced to the calculation of the two-time integral

$$\sigma(t_1,t_2)\equiv\frac{1}{2}\int_0^{t_1}\int_0^{t_2}e^{\gamma(s_1-t_1)}e^{\gamma(s_2-t_2)}\langle\xi(s_1)\xi(s_2)\rangle ds_1 ds_2,$$

which is the kernel needed to write the two-time joint probability distribution $P(x_1,t_1;x_2,t_2)$. See Appendix A for a specific example of this calculation in the simplest situation: the generalized strong non-Markovian Wiener particle.

We emphasize that in the presence of short-range correlated noise, the relaxation—after a transient—will be exponential. In this particular case, using the short-range correlation (2.1), the generalized dispersion gives

$$\sigma_{\rm short}(t)=\frac{\Gamma_2}{4\gamma}\left\{\frac{1-e^{-2\gamma t}}{(1-\theta^2\gamma^2)}+\frac{\gamma\theta(2e^{-(\gamma+(1/\theta))t}-e^{-2\gamma t}-1)}{(1-\theta^2\gamma^2)}\right\}. \tag{2.24}$$

This expression is valid at all times and for any $\theta\geqslant0$ and $\gamma\geqslant0$. Note that although both summands in Eq. (2.24) have non-Markovian contributions, the long-time limit is exponential. As a matter of fact, using Eq. (2.24) in Eq. (2.9) gives the exact one-time probability distribution at all times, for a particle in a harmonic well under the influence of a Gaussian color noise. In the limit $\theta\to0$ this expression coincides, of course, with the well known result that could be obtained from Fokker-Planck dynamics: $\sigma_{\rm white}(t)=(\Gamma_2/4\gamma)(1-e^{-2\gamma t})$.

III. REMARKS ON SECOND-ORDER SDE

A. Linear SDE with inertia (the rigid rotator)

An interesting application of the preceding sections is the study of a plane rotator [12,13] under the influence of long-range random torques. In this case the SDE is of second order,

$$\frac{d^2}{dt^2}\phi+\gamma\frac{d}{dt}\phi=\xi(t),\quad \phi\in(-\infty,\infty), \tag{3.1}$$

with an arbitrary noise $\xi(t)$. [A nonautonomous case could be if the random torques are time-periodically modulated [2]. Note that (3.1) can be trivially written in terms of variables $(\phi,\dot\phi)$, so the marginal SP ϕ will always be non-Markovian.] The exact generating functional of the SP $\phi(t)$, for $t\in[0,\infty]$, is

$$G_\phi([Z(t)])=\exp i(k_0\phi_0+q_0\dot\phi_0)\times G_\xi\left(\left[\int_t^\infty e^{\gamma(t-s)}\int_s^\infty Z(s')ds'ds\right]\right), \tag{3.2}$$

where $\phi_0\equiv\phi(0)$ and $\dot\phi_0\equiv\dot\phi(0)$ are the initial conditions of the rotator, and $k_0=\int_0^\infty Z(s)ds$ and $q_0=\int_0^\infty e^{-\gamma s}\int_s^\infty Z(s')ds'ds$. Formula (3.2) follows using propositions 1 and 3 of reference [2]. This result can also be seen putting $X(t)=(d/dt)\phi$ in Eq. (1.1), and then using Eq. (A2) with $\beta(t)=1$ and the assignation $W\to\phi$ and $\xi\to X$.

In formula (3.2) the generating functional $G_\xi([k(t)])$ characterizes an arbitrary noise $\xi(t)$. If we assume a Gaussian long-range noise we have to use Eqs. (2.2) and (1.2). Thus in this particular Gaussian case $G_\phi([Z(t)])$ adopts its simplest structure, which can immediately be used to calculate all the m-time moments of the SP $\phi(t)$, and in general any m-time correlation and cumulant of the stochastic angle $\phi(t)$.

A remarkable conclusion from Eq. (3.2) is that whenever that we have a closed expression for the functional of the noise $G_\xi([k(t)])$ (see Appendix B for non-Gaussian examples), nontrivial objects such as $\langle\cos\phi(t_1)\rangle$, $\langle\cos\phi(t_1)\cos\phi(t_2)\rangle$, etc. can be worked out—in an exact way—using trigonometric algebra. This is so because the functional of the stochastic cosine is given by $\langle\cos\int_0^\infty\phi(t)Z(t)dt\rangle=\mathcal{R}_e G_\phi([Z(t)])$. Then using Eq. (3.2), for the cosine functional we obtain the exact result

$$\left\langle\cos\int_0^\infty\phi(t)Z(t)dt\right\rangle=\mathcal{R}_e\left\{e^{i(k_0\phi_0+q_0\dot\phi_0)}G_\xi\left(\left[\int_t^\infty e^{\gamma(t-s)}\int_s^\infty Z(s')ds'ds\right]\right)\right\}. \tag{3.3}$$

We remark that this result is valid for any noise structure. Assuming a Gaussian long-range noise, for the mean value of the cosine we obtain the result

$$\langle\cos\phi(t_1)\rangle=\mathcal{R}_e\{G_\phi([Z(t)])\}_{Z(t)=\delta(t-t_1)}=\cos\left[\phi_0+\frac{1}{\gamma}(1-e^{-\gamma t_1})\dot\phi_0\right]\exp\left[-\frac{1}{2}\sum(t_1)\right], \tag{3.4}$$

where $\Sigma(t)$ is given by

$$\sum(t)=\int_0^t ds_1\int_0^t ds_2\int_{s_1}^t e^{\gamma(s_1-t')}dt'\times\int_{s_2}^t e^{\gamma(s_2-t^*)}dt^*\langle\xi(s_1)\xi(s_2)\rangle. \tag{3.5}$$

From Eq. (3.5), and using Eq. (1.2), it is possible to see that a strong non-Markovian behavior is found only if $\mu<1$, (logarithmic corrections happen for the case $\mu=1$). Hence in the long-time limit we obtain

$$\sum(t)\approx\frac{2\Gamma_2\tau^{\mu-1}\gamma^{-2}}{(2-\mu)(1-\mu)}t^{2-\mu},\quad t\gg\tau,\mu\in[0,1),\quad(3.6)$$

predicting an asymptotic anomalous dynamics $\langle\cos\phi(t)\rangle\approx\exp(-[\Gamma_2\tau^{\mu-1}\gamma^{-2}/(2-\mu)(1-\mu)]t^{2-\mu})$. Only if $\mu>1$, or if the correlation function of the noise is of the short-range type, will the long-time cosine relaxation be exponential $\langle\cos\phi(t)\rangle\approx\exp(-Ct)$, with $C=\text{const}$.

The cosine-cosine correlation is another important object which can be studied considering the average $\langle\cos(\phi(t_1)-\phi(t_2))\rangle$. In order to evaluate this mean value we can use the cosine functional with the test function $Z(t)=\delta(t-t_1)-\delta(t-t_2)$. Then using Eq. (3.3) we obtain the expression

$$\langle\cos(\phi(t_1)-\phi(t_2))\rangle=\mathcal{R}_e\left\{\exp\left(\frac{i}{\gamma}\left(e^{-\gamma t_2}-e^{-\gamma t_1}\right)\dot{\phi}_0\right)\times G_\xi\left(\left[\theta(t_1-t)\frac{e^{\gamma(t-t_1)}}{\gamma}-\theta(t_2-t)\frac{e^{\gamma(t-t_2)}}{\gamma}\right]\right)\right\};\quad(3.7)$$

here $\theta(t)$ is the step function. We note that Eq. (3.7) is a result valid for any noise structure. For the particular case of a Gaussian long-range noise, using Eqs. (2.2) and (1.2) we can obtain an exact expression for all time. It is interesting to study its asymptotic behavior, as it is then possible to see that, in the long-time limit, when $\mu<1$, a strong non-Markovian dynamics is obtained:

$$\langle\cos(\phi(t_1)-\phi(t_2))\rangle\approx\cos\left(\frac{\dot{\phi}_0}{\gamma}(e^{-\gamma t_2}-e^{-\gamma t_1})\right)\exp\left(-\frac{\Gamma_2\tau^{\mu-1}\gamma^{-2}}{(1-\mu)(2-\mu)}|t_1-t_2|^{2-\mu}\right),\quad t_1,t_2\gg\tau,\quad\mu\in[0,1).\quad(3.8)$$

Only if $\mu>1$, or if the correlation function of the noise is of the short-range type, will the cosine-cosine correlation (in the long-time regime) be exponential:

$$\langle\cos(\phi(t_1)-\phi(t_2))\rangle\approx\cos\left(\frac{\dot{\phi}_0}{\gamma}(e^{-\gamma t_2}-e^{-\gamma t_1})\right)\exp\left(+\frac{\Gamma_2\gamma^{-2}}{(1-\mu)}|t_1-t_2|\right),\quad t_1,t_2\gg\tau,\quad\mu>1,\quad(3.9)$$

i.e., we obtain the same dynamics as when using white noise.

B. Undamped free particle under arbitrary random accelerations

Here we show that our functional approach can also be used to study the influence of non-Gaussian noises with or without long-range correlations. To fix ideas, let the second order SDE be the one associated to an undamped free particle in presence of an arbitrary random force [in general with $\langle\xi(t)\rangle\neq0$],

$$\frac{d^2}{dt^2}Y(t)=\xi(t),\quad Y\in(-\infty,\infty),\quad(3.10)$$

so that the random acceleration is an arbitrary noise (Gaussian or not) characterized by its functional $G_\xi([k(t)])$; many non-Gaussian noise functionals can be found in Refs. [1,2,11,17,18], for example, radioactive noise, Campbell noise, dichotomous noise, Lévy noise [10], etc. (see Appendix B). Therefore, the complete characterization of the SP $Y(t)$ can be given in terms of such noise functionals. Following steps similar to those we made in Sec. III A, for the functional of the SP $Y(t)$ we obtain

$$G_Y([Z(t)])=\exp i(k_0Y_0+q_0\dot{Y}_0)\times G_\xi\left(\left[\int_t^\infty\int_s^\infty Z(s')ds'ds\right]\right),\quad(3.11)$$

where $Y_0\equiv Y(0)$ and $\dot{Y}_0\equiv\dot{Y}(0)$ are the initial conditions of the free particle, and $k_0=\int_0^\infty Z(s)ds$ and $q_0=\int_0^\infty\int_s^\infty Z(s')ds'ds$. Now we focus on a Gaussian long-range noise, and we will show the occurrence of strong non-Markovian results; non-Gaussian examples can be found in Appendix B. Hence for the Gaussian noise case we obtain that the one-time characteristic function of SP $Y(t)$ is given by

$$G_Y(k_1,t_1)=G_Y([Z(t)=k_1\delta(t-t_1)])=e^{i(k_1Y_0+k_1t_1\dot{Y}_0)}\exp\left(-\frac{1}{2}k_1^2\sum(t_1)\right),\quad(3.12)$$

where $\Sigma(t)$ can be read off from Eq. (3.5), taking the limit $\gamma=0$, and we can write

$$\sum(t)=2\int_0^t ds_1\int_0^{s_1}ds_2(t-s_1)(t-s_2)\langle\xi(s_1)\xi(s_2)\rangle.\quad(3.13)$$

Using Eq. (1.2) in Eq. (3.13) we obtain (for $\mu\neq1$, 2, 3, and 4)

$$\sum(t)=2\Gamma_2\tau^{-1}\left\{\frac{\tau t^3}{3(\mu-1)}-\frac{\tau^2t^2}{2(\mu-1)(\mu-2)}+\frac{\tau^4}{(\mu-1)(\mu-2)(\mu-3)(\mu-4)}\right.$$
$$\left.+\left(\frac{\tau}{t+\tau}\right)^{\mu}\frac{[(3-\mu)t^4+(8-3\mu)\tau t^3+(6-3\mu)\tau^2t^2-\mu\tau^3t-\tau^4]}{(\mu-1)(\mu-2)(\mu-3)(\mu-4)}\right\}. \quad (3.14)$$

[Note that we do not use the Tauberian theorem (as was used in Ref. [9]) to calculate $\Sigma(t)$; therefore, we can analyze any value of $\mu>0$.]

From this result it is simple to see that if $\mu<1$, the asymptotic (long-time limit) of the variance $\Sigma(t)$ is anomalous:

$$\sum(t)\approx\frac{2\Gamma_2\tau^{\mu-1}t^{4-\mu}}{(4-\mu)(2-\mu)(1-\mu)},\quad t\gg\tau,\mu\in[0,1). \quad (3.15)$$

The case $\mu=0$ gives the ballistic limit, and the case $\mu=1$ gives logarithmic corrections. Note from Eq. (3.13) that the Gaussian white-noise case (undamped free particle) gives the result $\Sigma(t)=1/3\Gamma_2t^3$. From Eq. (3.14) it is possible to see that if $\mu>1$, a weak non-Markovian behavior occurs, i.e., after the transient the following asymptotic behavior is reached:

$$\sum(t)\approx\frac{2}{3(\mu-1)}\Gamma_2t^3,\quad t\gg\tau,\mu>1, \quad (3.16)$$

as in the white-noise case.

From Eq. (3.12) it is also possible to obtain the asymptotic scaling of the one-time characteristic function; therefore, the scaling structure of the SP $Y(t)$ can be analyzed. Using Eq. (3.12) with $Y_0=\dot{Y}_0=0$, and Eq. (3.15), we obtain the following asymptotic scaling:

$$G_Y\left(\frac{k}{\sqrt{\Lambda^{4-\mu}}},\Lambda t\right)\rightarrow G_Y(k,t),\quad t\gg\tau,\mu\in[0,1). \quad (3.17)$$

Nevertheless, as noted above, if $\mu>1$ a weak non-Markovian behavior is obtained; hence its asymptotic scaling is the same as when using a white-noise $\xi(t)$, i.e.,

$$G_Y\left(\frac{k}{\sqrt{\Lambda^3}},\Lambda t\right)\rightarrow G_Y(k,t). \quad (3.18)$$

Using Eq. (3.17), we can conclude, for $\mu\in[0,1)$, that the dynamics of the strong non-Markovian SP $Y(t)$ is different compared with the case where the driving noise $\xi(t)$ is white. The spectrum of the SP $Y(t)$ behaves asymptotically like

$$S_Y(f)\propto\frac{1}{f^{5-\mu}},\quad t\gg\tau,\mu\in[0,1). \quad (3.19)$$

Of course, in the weak non-Markovian case, $\mu>1$, we asymptotically reobtain $S_Y(f)\propto1/f^4$, as expected. We note that the box counter fractal dimension [10] of the SP $Y(t)$ would give asymptotically

$$D_B=\frac{\mu}{2},\quad t\gg\tau,\mu\in[0,1), \quad (3.20)$$

and the divider fractal dimension along the curve $Y(t)$ would give asymptotically

$$D=\frac{2}{4-\mu},\quad t\gg\tau,\mu\in[0,1). \quad (3.21)$$

But we cannot take (3.20) and (3.21) as correct fractal dimensions because they are less than one. So the fractal dimensions equal the topological dimension.

IV. GENERAL CONCLUSIONS

The strong non-Markovian harmonic potential process $X(t)$ has been completely characterized in terms of its exact functional $G_X([Z(t)])$; see Eq. (2.5) for the Gaussian case. Particular stress has been put on long-range correlated noise where $\langle\xi(t_1)\xi(t_2)\rangle$ is of the form given in Eq. (1.2). This fairly general method is based upon knowing the characteristic functional of the noise $G_\xi([k(t)])$, which in the present paper has mainly been assumed to be Gaussian (non-Gaussian statistics will be worked out in Appendix B). Thus any m-time moment of the SP $X(t)$ follows straightforwardly by taking the functional derivative of $G_X([Z(t)])$; see Eq. (2.6).

We have given an exact expression for the stationary (nonequilibrium) probability distribution of a particle in a harmonic potential and in the presence of an arbitrary correlated Gaussian noise; see Eq. (2.14). There the generalized stationary dispersion σ_∞ has been calculated explicitly [see Eq. (2.13)] for two nontrivial cases: the color-noise case (2.15) (weak non-Markovian effects), and the long-range case (2.17) (strong non-Markovian effects).

Our physical motivation to study this type of linear SDE was the model of a rigid rotator in the presence of arbitrary Gaussian long-range random torques. This problem was worked out in Sec. III A, and its solution, for any noise $\xi(t)$, was given in Eq. (3.2). In particular, the asymptotic behavior of the cosine relaxation and the cosine-cosine correlation have been calculated for the Gaussian long-range case. The undamped free particle has been studied in order to analyze its scaling structure as a function of the exponent of the power law appearing in the long-range correlated noise. To show other applications of our functional technique, the one-time characteristic function of the undamped free particle has explicitly been calculated—in an exact way—for several non-Gaussian noises (see Appendix B).

The present formulation is exact, and provides a systematic starting point to obtain higher-order moments or to compute the whole Kolmogorov hierarchy for any structure of the noise; see Eq. (2.7). The two-time correlation function $\langle\langle W(t_1)W(t_2)\rangle\rangle$ for the generalized strong non-Markovian Wiener particle has been calculated to show the persistence of its generalized increments; see Appendix A.

We note that there are no limitations in the exact calculation of any higher-order moment, etc. In the present paper we have used a functional technique to solve a problem with natural boundary conditions; the application of this method to a problem with a non-natural boundary condition [11] is under investigation.

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APPENDIX A: THE TWO-TIME CUMULANT OF A GAUSSIAN STRONG NON-MARKOVIAN PROCESS

In order to have a more comprehensive idea of what a long-range correlated noise produces in a SDE, let us present here the generalized strong non-Markovian Wiener particle. Let the evolution equation of the displacement of a particle be characterized by the nonautonomous SDE

$$\frac{d}{dt}W=\beta(t)\xi(t), \quad W\in(-\infty,\infty), \tag{A1}$$

with $\beta(t)$ a sure function of time and $\xi(t)$ an arbitrary noise. Using proposition 3 of Ref. [2], the characteristic functional of the SP $W(t)$, for any noise $\xi(t)$, characterized by its functional $G_\xi([k(t)])$, is given by

$$G_W([Z(t)])=e^{+ik_0W_0}G_\xi\left(\left[\beta(t)\int_t^\infty Z(s)ds\right]\right), \tag{A2}$$

where $k_0=\int_0^\infty Z(s)ds$, and $W_0\equiv W(0)$ is the initial condition.

For the Gaussian case $G_\xi([k(t)])$ is given in Eq. (2.2), and $\langle\xi(t_1)\xi(t_2)\rangle$ can be any correlation, for example the long-range case (1.2). Let us now study, in the autonomous case $\beta(t)=1$, the correlation function of the displacement of the particle $\langle\langle W(t_1)W(t_2)\rangle\rangle$; this function can easily be calculated by using functional derivatives as in Eq. (2.6):

$$\begin{aligned}\langle\langle W(t_1)W(t_2)\rangle\rangle&=i^{-2}\left.\frac{\delta^2\ln G_W([Z(t)])}{\delta Z(t_1)\,\delta Z(t_2)}\right|_{Z=0}\\&=i^{-2}\frac{\delta^2}{\delta Z(t_1)\,\delta Z(t_2)}\Bigg[iW_0\int_0^\infty Z(s)ds\\&\quad-\frac{1}{2}\int_0^\infty\int_0^\infty\left(\int_{s_1}^\infty Z(s)ds\right)\\&\quad\times\left(\int_{s_2}^\infty Z(s')ds'\right)\\&\quad\times\langle\xi(s_1)\xi(s_2)\rangle ds_1ds_2\Bigg]_{Z=0}.\end{aligned} \tag{A3}$$

Without loss of generality, we now assume that $t_1\leqslant t_2$; then we obtain

$$\begin{aligned}\langle\langle W(t_1)W(t_2)\rangle\rangle&=\int_0^{t_1}\int_0^{t_2}\langle\xi(s_1)\xi(s_2)\rangle ds_1ds_2\\&=2\sigma(t_1)+\int_0^{t_1}ds_1\int_{t_1}^{t_2}ds_2\langle\xi(s_1)\xi(s_2)\rangle.\end{aligned} \tag{A4}$$

For the particular case of noise $\xi(t)$ with long-range correlation, like that in Eq. (1.2), the result (for $\mu\neq\{1,2\}$) is

$$\begin{aligned}\frac{1}{\Gamma_2}\langle\langle W(t_1)W(t_2)\rangle\rangle&=\frac{2t_1}{(\mu-1)}-\frac{\tau}{(\mu-1)(\mu-2)}\\&\quad+\frac{\tau^{\mu-1}}{(\mu-1)(\mu-2)}[(t_1+\tau)^{2-\mu}\\&\quad+(t_2+\tau)^{2-\mu}-(t_2-t_1+\tau)^{2-\mu}].\end{aligned} \tag{A5}$$

In the limits $t_1\gg\tau$ and $t_2\gg\tau$ such that $t_2-t_1\sim O(\tau)$, and if the noise parameter $\mu\in[0,1)$, we see that the two-point second cumulant of the SP $W(t)$ increases with time with a power law $\sim(t_1^{2-\mu}+t_2^{2-\mu})$ (i.e., superdiffusion). Thus a long-range correlated noise induces a nonlinear temporal behavior in $\langle\langle W(t_1)W(t_2)\rangle\rangle$ as in the fBm for $H\in(1/2,1]$. Note also that both times are present in this nonstationary correlation function. If the noise parameter is $\mu>1$, we reobtain, in the asymptotic long-time regime, the Wiener result $\langle\langle W(t_1)W(t_2)\rangle\rangle\propto\min(t_1,t_2)$.

From the exact result (A5) it is possible to see that, in the long-time regime, if $\mu\in[0,1)$, the increments of the SP $W(t)$ are not statistically independent. The case $\mu=1$ also shows non-Markovian effects with logarithmic corrections. To be more precise, let us calculate here a normalized correlation function of future increments $[W(t)-W(0)]$ with past increments $[W(0)-W(-t)]$:

$$\begin{aligned}C(t)&\equiv\frac{1}{\langle W(t)^2\rangle}\langle[W(0)-W(-t)][W(t)-W(0)]\rangle\\&=\frac{1}{\langle W(t)^2\rangle}\{\langle W(0)W(t)\rangle-\langle W(0)^2\rangle-\langle W(-t)W(t)\rangle\\&\quad+\langle W(-t)W(0)\rangle\}.\end{aligned} \tag{A6}$$

From Eq. (A5) it is simple to show that (for $0<\mu\neq1,2$)

$$C(t)=\frac{\tau+\tau^{\mu-1}(\tau+2t)^{2-\mu}}{2(\mu-2)t-2\tau+2\tau^{\mu-1}(\tau+t)^{2-\mu}}. \tag{A7}$$

Therefore (in the asymptotic long-time limit), only if $0\leqslant\mu<1$ are past increments correlated with future increments, i.e., a long-range noise with $\mu\in[0,1)$ induces infinitely long-run correlations in the increments of the SP $W(t)$, as in the persistent fBm. Nevertheless, if $\mu>1$ (weak non-Markovian behavior) the normalized correlation function $C(t)$ goes to zero in the limit $t\to\infty$, and is therefore in agreement with a Wiener-like behavior.

It is also possible to see, using the fact that the noise $\xi(t)$ is symmetric and adopting the initial condition $W(0)\equiv W_0=0$ in Eq. (A3), that the variance of an arbitrary increment of our SP $W(t)$ is given by [for $0<\mu\neq 1,2$ and assuming $t_1\leqslant t_2$]

$$\frac{1}{\Gamma_2}\langle[W(t_2)-W(t_1)]^2\rangle=\frac{2}{(\mu-1)(\mu-2)}\,[(t_2-t_1)(\mu-2)-\tau+\tau^{\mu-1}(\tau+t_2-t_1)^{2-\mu}]. \tag{A8}$$

Thus for large t_2-t_1 such that $t_2-t_1\gg\tau$, and if the noise parameter $\mu\in[0,1)$, we see that $\langle[W(t_2)-W(t_1)]^2\rangle$ increases with time as $\sim(t_2-t_1)^{2-\mu}$, in agreement with the picture of a fBm, i.e., a long-range correlated noise $\xi(t)$ induces a power-law behavior in the variance of the increments of the SP $W(t)$. If the noise parameter is $\mu>1$ we reobtain—in the asymptotic long-time regime—a Wiener-like result: $\langle[W(t_2)-W(t_1)]^2\rangle\sim|t_2-t_1|$.

From Eq. (A5) and consistently from Eqs. (A7) and (A8), it is possible to see that the spectrum behaves asymptotically like $S_W(f)\propto 1/f^{3-\mu}$, the box counter fractal dimension of the record is $D_B=1+(\mu/2)$, and the fractal divider dimension along the curve $W(t)$ is $D=2/(2-\mu)$. See Refs. [4,10] to study this fractal analysis in terms of the asymptotic scaling $W(\Lambda t)\to\sqrt{\Lambda^{2-\mu}}W(t)$ for $t\gg\tau$ and $\mu\in[0,1)$.

APPENDIX B: UNDAMPED FREE PARTICLE UNDER NON-GAUSSIAN NOISES

Here we present a few results concerning non-Gaussian noise [2,11]. Let us assume, for example, that the source of noise in the SDE (3.10) is of the radioactive type; thus we use the corresponding noise's functional $G_\xi([k(t)])=[\beta\int_0^\infty \exp(-t\beta+i\int_0^t k(s)ds)dt]^{\xi_0}$ for $t\in[0,\infty)$. Here ξ_0 represents the number of active nuclei at $t=0$, and β is the probability per unit of time that an individual decay occurs; note that this noise is nonstationary. Then using Eq. (3.11), we obtain, for the one-time characteristic function of the SP $Y(t)$,

$$G_Y(k_1,t_1)=e^{i(k_1Y_0+k_1t_1\dot{Y}_0)}\times\left[\beta\int_0^{t_1}\exp\left((ik_1t_1-\beta)t-i\frac{k_1}{2}t^2\right)dt+\exp\left(i\frac{k_1}{2}t_1^2-\beta t_1\right)\right]^{\xi_0}, \tag{B1}$$

where Y_0 and $\dot{Y}_0$ are the initial conditions of the SP $Y(t)$. It is simple to see that Eq. (B1) fulfills normalization $G_Y(0,t)=1$, and, for example, the first moment is given by

$$\langle Y(t)\rangle=i^{-1}\frac{d}{dk}G_Y(k,t)|_{k=0}=(Y_0+t\dot{Y}_0)+\frac{\xi_0}{\beta^2}\,(\beta t-1+e^{-\beta t}). \tag{B2}$$

Note that the SP $Y(t)$ is also nonstationary, as expected, and due to the fact that the correlation of the noise is of the short-range class a nonanomalous behavior is (asymptotically) obtained for the SP $Y(t)$.

For a random force having the structure of a Campbell noise, we know that its functional is $G_\xi([Z(t)])=\exp\int_0^\infty(\exp[i\int_0^\infty k(t)\psi(t-\tau)dt]-1)q(\tau)d\tau$, for $t\in[0,\infty)$; here $q(\tau)$ is the density of one dot, and $\psi(t)$ is the shape of each pulse. Because the correlation function of this noise is $\langle\langle\xi(s_1)\xi(s_2)\rangle\rangle=\int_0^\infty q(\tau)\psi(s_1-\tau)\psi(s_2-\tau)d\tau$, we can expect a long-range noise if the shape of the pulse is a power-law function. Depending on the function $q(t)$, Campbell's noise could be nonstationary. Note that the mean value of the noise is just $\langle\xi(t)\rangle=\int_0^\infty q(\tau)\psi(t-\tau)d\tau$. Therefore, using a power law for the shape of $\psi(t)$ we will obtain a strong non-Markovian SP $Y(t)$, and in addition a non-Gaussian character. Using Eq. (3.11) for the one-time characteristic function of the SP $Y(t)$ we obtain

$$G_Y(k_1,t_1)=e^{i(k_1Y_0+k_1t_1\dot{Y}_0)}\exp\left\{\int_0^\infty\left(\exp\left[ik_1t_1\times\int_0^{t_1}\psi(t-\tau)dt\right]-1\right)q(\tau)d\tau\right\}. \tag{B3}$$

As with the previous example, it is simple to see that $G_Y(0,t_1)=1$, and in this case the first moment is given by

$$\langle Y(t_1)\rangle=(Y_0+t_1\dot{Y}_0)+t_1\int_0^\infty q(\tau)d\tau\int_0^{t_1}\psi(t-\tau)dt. \tag{B4}$$

Hence, as stated above depending on the function $\psi(t)$ an anomalous behavior (strong non-Markovian dynamics) will be found.

For a Lévy noise [10] the functional reads $G_\xi([Z(t)])=\exp(-b\int_0^\infty|k(s)|^\alpha ds)$, for $t\in[0,\infty)$, with $b>0$ and $\alpha\in(0,2]$. Using Eq. (3.11) for the one-time characteristic function of the SP $Y(t)$ we obtain

$$G_Y(k_1,t_1)=e^{i(k_1Y_0+k_1t_1\dot{Y}_0)}\exp\left(-b\frac{t_1^{\alpha+1}}{\alpha+1}|k_1|^\alpha\right),\quad \alpha\in(0,2]. \tag{B5}$$

As expected, the one-time probability distribution $P(Y[t_1])\equiv P(y_1,t_1)$ is momentless. Note the difference from the characteristic function of Lévy flights [10] $G_L(k,t)=e^{ikL(0)}\exp(-bt|k|^\alpha)$. The structure of this SP $Y(t)$ can also be analyzed by studying the scaling invariance ofthe one-time characteristic function; from Eq. (B5), with $Y_0=\dot{Y}_0=0$, we immediately see that

$$G_Y\left(\frac{k}{\Lambda^{1+(1/\alpha)}},\Lambda t\right)=G_Y(k,t). \tag{B6}$$

So the SP $Y(t)$ would lead to a smaller fractal divider dimension $[D=\alpha/(1+\alpha)]$ than the Lévy flights $(D=\alpha)$. But we cannot take that as a correct fractal dimension because it implies an absurd. Indeed, this dimension contradicts the property that the Hausdorff dimension is never less than the topological one. We end this section by pointing out that Eq. (B3) can be used as the starting point to study the interplay between non-Gaussian and long-range correlations.

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